CSIR UGC NET

MATHEMATICAL SCIENCE

SOLVED SAMPLE PAPER

* DETAILED SOLUTIONS

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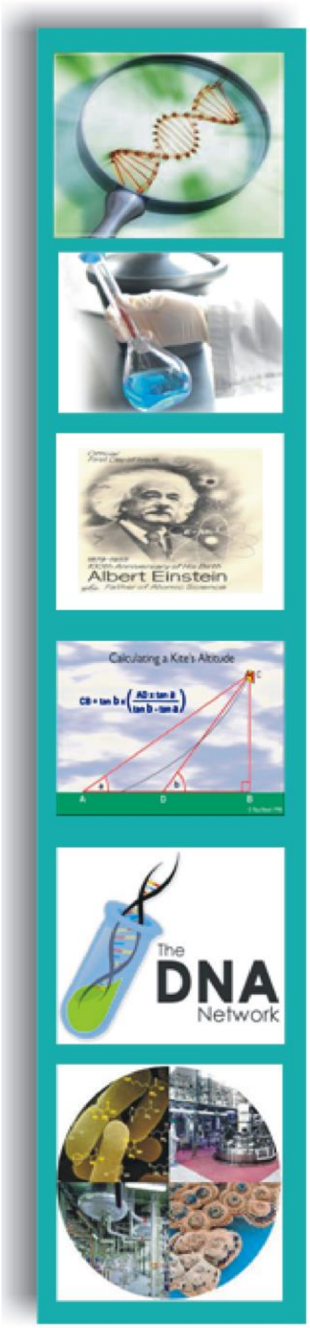

CSIR NET - MATHEMATICAL SCIENCE

MOCK TEST PAPER

- ***This paper contains 60 Multiple Choice Questions***
- ***part A 15, part B 25 and part C 20***
- ***Each question in Part 'A' carries two marks***
- ***Part 'B' carries 3 marks***
- ***Part 'C' carries 4.75 marks respectively.***
- ***There will be negative marking @ 0.5 marks in Part 'A' , 0.75 marks in Part 'B' for each wrong answer.***
- ***Part 'C' has more than 1 correct options***
- ***Pattern of questions : MCQs***
- ***Total marks : 200***
- ***Duration of test : 3 Hours***

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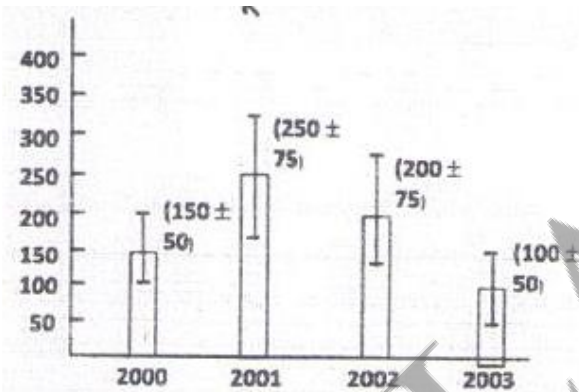
PART A (1-15)

1. Average yield of a product in different years is shown in the histogram. If the vertical bars indicate variability during the year, then during which year was the percent variability over the average of that year the least?



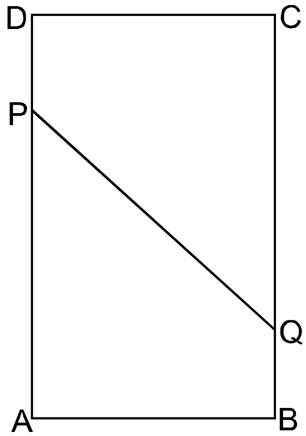
(1) 2000

(2) 2001

(3) 2002

(4) 2003

2. A rectangular sheet ABCD is folded in such a way that vertex A meets vertex C, thereby forming a line PQ. Assuming AB = 3 and BC = 4, find PQ. Note that AP = PC and AQ = QC.



(1) $\frac{13}{4}$

(2) $\frac{15}{4}$

(3) $\frac{17}{4}$

(4) $\frac{19}{4}$

3. Density of a rice grain is 1.5 g/cc and bulk density of rice heap is 0.80 g/cc. If a 1 litre container is completely filled with rice, what will be the approximate volume of pore space in the container?

(1) 350 cc

(2) 465 cc

(3) 550 cc

(4) 665 cc

4. A peacock perched on the top of a 12 m high tree spots a snake moving towards its hole at the base of the tree from a distance equal to thrice the height of the tree. The peacock flies towards the snake in a straight line and they both move at the same speed. At what distance from the base of the tree will the peacock catch the snake?

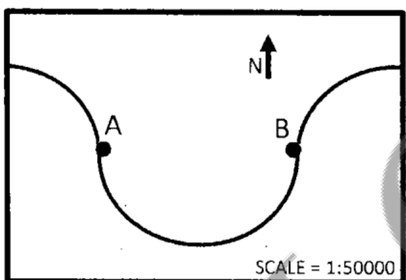
(1) 16 m

(2) 18 m

(3) 14 m

(4) 12 m

5. The map given below shows a meandering river following a semi-circular path, along which two villages are located at A and B. The distance between A and B along the east-west direction in the map is 7 cm. What is the length of the river between A and B in the ground?



(1) 1.1 km

(2) 3.5 km

(3) 5.5 km

(4) 11.0 km

6. How many nine-digit positive integers are there, the sum of squares of whose digits is 2?

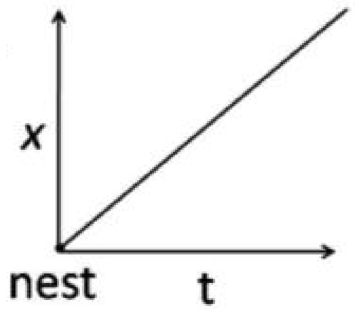
(1) 8

(2) 9

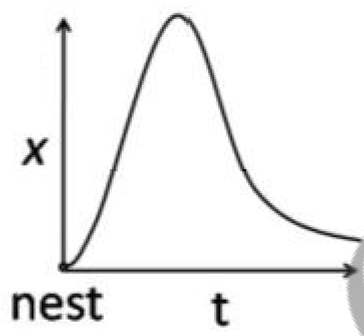
(3) 10

(D) 11

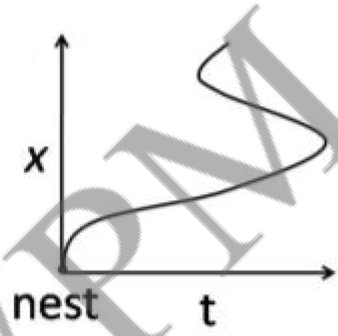
7. A bird leaves its nest and flies away. Its distance x from the nest is plotted as a function of time t. Which of the following plots cannot be right?



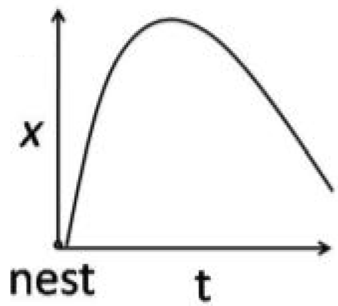
(1)



(2)



(3)

(4) 

8. What is the next number in the following sequence?

39, 42, 46, 50, ...

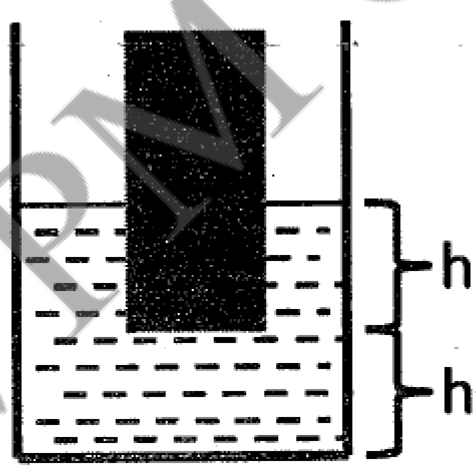
(1) 52

(2) 53

(3) 54

(D) 55

9. A solid cylinder of basal area A was held dipped in water in a cylindrical vessel of basal area 2A vertically such that a length h of the cylinder is immersed. The lower tip of the cylinder is at a height h from the water in the vessel when the cylinder is taken out?



(1) 2h

(2) $\frac{3}{2}h$

(3) $\frac{4}{3}h$

(4) $\frac{5}{4}h$

10. How many pairs of positive integers have gcd 20 and lcm 600?

(gcd = greatest common divisor; lcm = least common multiple)

(1) 4

(2) 0

(3) 1

(4) 7

11. Consider a right-angled triangle ABC where AB = AC = 3. A rectangle APOQ is drawn inside it, as shown, such that the height of the rectangle is twice its width. The rectangle is moved horizontally by a distance 0.2 as shown schematically in the diagram (not to scale).

What is the value of the ratio $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta OST}$?

(1) 625

(2) 400

(3) 225

(4) 125

12. A shopkeeper purchases a product for Rs. 100 and sells it making a profit of 10%. The customer resells it to the same shopkeeper incurring a loss of 10%. In these dealings the shopkeeper makes

(1) no profit, no loss

(2) Rs. 11

(3) Re. 1

(4) Rs. 20

13. In 450 g of pure coffee powder 50 g of chicory is added. A person buys 100 g of this mixture and adds 5 g of chicory to that. What would be the rounded-off percentage of chicory in this final mixture?

(1) 10

(2) 5

(3) 14

(4) 15

14. Following table provides figures (in rupees) on annual expenditure of a firm for two years - 2010 and 2011.

Category	2010	2011
Raw material	5200	6240
Power & fuel	7000	9450
Salary & wages	9000	12600
Plant & machinery	20000	25000
Advertising	15000	19500
Research & Development	22000	26400

In 2011, which of the following two categories have registered increase by same percentage?

(1) Raw material and Salary & wages

(2) Salary & wages and Advertising

(3) Power & fuel and Advertising

(4) Raw material and Research & Development

15. Find the missing sequence in the letter series.

B, FH, LNP, --------.

(1) SUMY

(2) TUVW

(3) TVXZ

(4) TWXZ

PART B(16-40)

16. Suppose a population A has 100 observations 101, 102, ... 200 and another population B has 100 observations 151, 152, ..., 250. If V_A and V_B represent the variances of the two populations, respectively, then $\frac{V_A}{V_B}$ is

(1) 1

(2) $\frac{9}{4}$

(3) $\frac{4}{9}$

(4) $\frac{2}{3}$

17. Let $y_1 < y_2 < y_3 < y_4$ denote the order statistics of a random sample of size 4 from a distribution having Pdf

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

then $P(y_3 > \frac{1}{2})$ equals

(1) $\frac{243}{512}$

(2) $\frac{729}{512}$

(3) $\frac{243}{256}$

(4) $\frac{81}{64}$

18. Let $T : V \to W$ and $S : W \to \Omega$ be two linear transformations then which one of the following is the false statement?

(1) If S and T one one-one onto then ST is one-one onto and $(ST)^{-1} = T^{-1}S^{-1}$

(2) If ST is one -one then T is one-one

(3) If ST is onto then S is onto

(4) If ST is onto then T is onto

19. If $Z_1 = 3 - 4i$ and $Z_2 = -4 + 3i$ then angle between Z_1 Z_2 is given by,

(1) $\cos^{-1}0.96$

(2) $\pi - \cos^{-1}0.96$

(3) $\cos^{-1}0.47$

(4) $\pi - \cos^{-1}0.47$

20. $I = \int_\gamma x\,dz$ where γ is the boundary of the square [0, 1] × [0, 1] with c considered as R^2 is given by

(1) 1

(2) 0

(3) $2\pi i$

(4) i

21. Which of the following statement is incorrect about bilinear transformation?

(1) The inverse of bilinear transformation is bilinear transformation

(2) Composition of bilinear transformation is bilinear transformation

(3) A bilinear transformation which fixes 1 is identity transformation

(4) Every bilinear transformation maps circles into circles

22. The function f : □ → □ defined by f(x) = $(x^2+1)^{35}$ for all real x ∈ □ is,

(1) one-one but not onto

(2) onto but not one-one

(3) neither one-one nor onto

(3) both one-one and onto

23. The complete integral of $q = 3p^2$ is given by,

(1) $z = ax + 3a^2 + c$

(2) $z = ax + a^2 + 3$

(3) $z^2 = a^2 ax$

(4) $z = ax + b$

24. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^P + n^{-P}}$, then which of the following is/are incorrect ?

(1) Convergent if P>1

(2) Divergent if $P \leq 1$

(3) Convergent if P<-1

(4) Divergent if $-1 \leq P \leq 1$

25. If $2^n - 1$ is prime for n >1 then n is,

(1) a prime

(2) a composite

(3) any natural number

(4) only odd prime

26. The order of smallest non-commutative ring is

(1) 1

(2) 2

(3) 3

(4) 4

27. The Hermite's interpolation polynomial which fits the data

x	0	4
f = f(x)	0	2
y' = f'(x)	1	0

is

(1) $\frac{1}{8}(8x - x^2)$

(2) $\frac{1}{16}(2x - x^2)$

(3) $(2x - x^2)$

(4) $(16x - 2x^2)$

28. Find the equation of curve fixed between two point $\left(0, \frac{1}{3}\right)\left(\frac{\pi}{2}\ \frac{1}{3}\right)$ where integral $I = \int_0^{\pi/2}\left(y'^2 - y^2 + 4y\sin^2 x\right)dx$

(1) y = (2 sinx + cos 2x)/3

(2) y = (2sinx + sin2x)/3

(3) y = (2cosx + cos 2x)/3

(4) y = (cosx + sin2x)/3

29. Let V be the space of all real valued continuous functions. Define T ; V → V by (Tf) (x) = $\int_0^x f(t)dt$

then find the eigen values of T.

(1) 0

(2) 1

(3) C (any arbitrary natural no.)

(4) Does not exist.

30. When $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta$ cos h $(2\cos\theta)d\theta$

(1) $\cosh\left(z + \frac{1}{z}\right) = a_0$

(2) $\cosh\left(z + \frac{1}{z}\right) = \sum_{n=1}^{\infty} a_n$

(3) $\cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

(d) $\cosh\left(z + \frac{1}{z}\right) = 0$

31. There are 600 business students in the post-graduate department of a university, and probability for any student to need a copy of a particular textbook from the university library on any day is 0.05. How many copies of the book should be kept in the university library so that the probability may be greater than 0.90 that none of the students needing a copy from the library has to come back disappointed. (Use normal approximation to the binomial probability law).

(1) 73

(2) 30

(3) 37

(4) 80

32. Consider the power series $\sum a_n z^{n^4}$, where $a_0 = 1$ and $a_n = a_{n-1}4^{-n^3}$, $n \geq 1$

(1) Radius of Convergent is 4

(2) Radius of Convergent is 2

(3) Radius of Convergent is $\sqrt{2}$

(4) Radius of Convergent is 1

33. Let $9(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$ and $f''(x) < 0 \ \forall x \in (0,2)$. Then g(x) increases in,

(1) $\left(\frac{1}{2}, 2\right)$

(2) $\left(\frac{4}{3}, 2\right)$

(3) (0,2)

(4) $\left(0, \frac{4}{3}\right)$

34. Divergence Criteria states that

(1) If a sequence X = $\{x_n\}$ of real number has two convergent subsequences X' = $\{x_{r_k}\}$ and X" = $\{x_{r_k}\}$ whose limits are equal then X is divergent.

(2) If a sequence X = $\{x_n\}$ of real number is bounded then X is divergent.

(3) If a sequence X = $\{x_n\}$ of real numbers has two convergent Subsequence X' = $\{x_{r_k}\}$ and X" = $\{x_{r_k}\}$ whose limits are not equal then X is divergent.

(4) If a sequence X = $\{x_n\}$ of real numbers is unbounded then X is convergent

35. Let I be an interval and let f : I $\rightarrow$ R be strictly monotone on I. Let J : = f(I) and let g : J $\rightarrow$ R be the function inverse to f . If f is differentiable on I and f' (x) $\neq$ 0 for x $\leq$ I, then g is differentiable on J and g' = $\frac{1}{f' o\, g}$

then,

(1) g' = $\frac{1}{g}$

(2) $g' = \frac{1}{fog'}$

(3) $g' = \frac{1}{f'og}$

(4) $g' = \frac{1}{fog}$

36. Find the quadratic equation eq^n in λ whose roots are the Eigen values of the integral equation

$$g(x) = \lambda \int_0^1 (2xt - 4x^2)\, g(t)\, dt$$

(1) $\lambda^2 + 6\lambda\ 9 = 0$

(2) $\lambda^2 - 6\lambda + 9 = 0$

(3) $\lambda^2 - 6\lambda - 9 = 0$

(4) $\lambda^2 + 6\lambda - 9 = 0$

37. Find the Resolvent kernal of the integral equation

$$g(x) = x + \int_0^{1/2} g(x)dt$$

(1) 0

(2) 1

(3) 2

(4) x

38. A uniform rod AB of length 8a is suspended from a fixed point O by means of light inexcusable string, of length 13a, attached to B. If the system is slightly displaced in a vertical plane the lagrange's θ-equation is

(1) $61\ddot{\theta} + 39\ddot{\phi} = -\frac{3g}{a}\theta$

(2) $61\ddot{\theta} + 39\ddot{\phi} = \frac{3g}{a}\theta$

(3) $4\ddot{\theta} + 13\ddot{\phi} = -\frac{g}{a}\theta$

(4) $4\ddot{\theta} + 13\ddot{\phi} = \frac{g}{a}\theta$

39. The modified Newton – Raphson's method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

given

(1) a non - quadratic convergence when the equation f(x) = 0 has a pair of double roots in the neighbourhood of x = x_n.

(2) a quadratic convergence when the equation f(x) = 0 has a pair of double roots in the neighborhood of x = x_n.

(3) a nonquadratic non-convergence.

(4) None of the above.

40. Let T be a linear operator on C^2 defined by $T(x_1, x_2) = (x_1, 0)$ Let $\beta = \{\epsilon_1 = (1, 0), \epsilon_2 = (0, 1)$, $\beta' = \{\alpha_1 = (1, i), \alpha_2 = (-i, 2)\}$ be ordered basis for C^2. What is the matrix of T relative to the pair β, β'?

(1) $\begin{bmatrix} 0 & 2 \\ 0 & -i \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 0 & 0 \\ 2 & -i \end{bmatrix}$

(4) $\begin{bmatrix} 2 & -i \\ 0 & 0 \end{bmatrix}$

PART C(41-60)

41. Let R_∞ be the extended set of real numbers the function d defined by

$$d(x, y) = |f(x) - f(y)| \quad \forall\, x,y \in R_\infty$$

where f(x) is given by

$$f(x) = \begin{cases} \dfrac{x}{1+|x|} & \text{when } -\infty < x < \infty \\ 1 & \text{when } x = \infty \\ -1 & \text{when } x = -\infty \end{cases}$$

Then–

(1) (R_∞ d) is metric space

(2) (R_∞ d) is bounded

(3) diameter of (R_∞ d) is 2

(4) R_∞ does not include ∞ or $-\infty$

42. Every bilinear transformation maps,

(1) circles into circles

(2) circles into lines

(3) lines into lines

(4) lines into circles

43. When interval of differencing is unity, then

(1) $\Delta\left(\dfrac{2}{x+2}\right) = 2\left\{\dfrac{1}{x+3} - \dfrac{1}{x+2}\right\}$

(2) $\Delta\left(\dfrac{3}{x+2}\right) = 3\left\{\dfrac{1}{x+4} - \dfrac{1}{x+3}\right\}$

(3) $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)=\frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$

(4) $\Delta^2 x = x(\Delta x)$

44. Let a_1, a_2, a_3, a_4, ...are integers such that

(I) $|a_n| < |a_{n+1}|\ \forall n \in$ □

(II) a_n divides $a_{n+1}, \forall n \in$ □

(III) Some of integers divides each a_n, then which of the following(s) is/are not true?

(1) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ cannot be convergent

(2) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ converge absolutely

(3) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is convergent

(4) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ converge conditionally

45. Let T be a linear operator on V and Let Rank T^2 = Rank T then

(1) nullity T^2 = nullity T

(2) Range $T \cap \text{Ker}T^2 = \{0\}$

(3) Range $T \cap \text{Ket}T = \{0\}$

(4) Range $T^2 \cap \text{Ker}\,T^2 = \{0\}$

46. Which of the following as a linear transformation on □ ^{2}?

(1) $T(x_1, x_2) = (x_2, x_1)$

(2) $T(x_1, x_2) = (\sin x_1, x_2)$

(3) $T(x_1, x_2) = (x_1^2, x_2)$

(4) $T(x_1, x_2) = (x_1 - x_2, 0)$

47. $\Sigma u_n(x)$ is a series of real valued functions defined as $u_1(x) = x$ $u_n(x) = x^{1/2n-1} - x^{1/2n-3}$ n =2, 3.... then $\Sigma u_n(x)$

(1) discontinuous

(2) non-uniformly convergent

(3) continuous

(4) can be integrated term by term

48. If f(z) is analytic in any domain D any function g(z) defined as g(z) = $\overline{f}(\overline{z})$ is

(1) analytic everywhere

(2) analytic in D

(3) analytic in D* = $\{z : \overline{z} \in D\}$

(4) if f'(z) = 0 in D then f(z) is free from z

49. If f(z) is integrable along a curve c having finite length ℓ and if there exists a positive number M such that $|f(z)| \le M$ on c then

(1) $\left|\int_c f(z)dz\right| = \text{constant}$

(2) $\left|\int_c f(z)dz\right| = 0$

(3) $\left|\int_c f(z)dz\right| \le \ell M$

(4) $\left|\int_c f(z)dz\right| < \infty$

50. $\dfrac{Z_5[z]}{\langle x^2 + 2x + 2\rangle}$ is

(1) A field having 32 elements

(2) a field having 25 elements

(3) a field having exactly 2 subfields

(4) isomorphic to $\frac{Z_5[z]}{\langle x^2 - 2x + 15 \rangle}$

51. Let $P(x) \in \square[x]$ then $\frac{\square[x]}{<P(x)>}$ be a field if ,

(1) P(x) = x^2+1

(2) P(x) = x^2-1

(3) P(x) = x^3-x^2+x-1

(4) P(x) = x^2+x+1

52. If $P_3(x) = x^3 - 5x^2 + 17x - 3$ be on three-degree polynomial

then if $\delta = \max_{0 \le x \le 4} |P_3(x) - P_2(x)|$

where $P_2(x)$ is second degree polynomial

Then

(1) $\delta = 2$

(2) for x = 0, 1, 4; $\delta = 2$

(3) for x = 3 δ = does not exist,

(4) for x= 0 , $\delta = 2$

53. Let $A = [a_{ij}]_{n \times n}$ be a matrix such that rows and columns of A forms an orthonormal set

Then possible cases/case are

(1) $a_{ij} \in$ **C** and A is unitary

(2) $a_{ij} \in$ **R** and A is orthogonal

(3) aij $\in$ **C** and A is orthogonal

(4) $a_{ij} \in$ **R** and A is unitary

54. The Given differential equation

$\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)-\frac{n^2}{x^2}\right]u=0$ with u (0) = 0 and u (1) = 0 have

(1) Linearly independent solution

(2) Green function defined on it

(3) $W'[u_1(x), u_2(x)] = 0$

(4) $G(x, \xi) = 0\ \forall x$

55. The functional $\int_0^{\pi/2}(y'^2 - y^2 + 2xy)dy$ with y(0) = 0 and $y\left(\frac{\pi}{2}\right)=0$ is extremized by

(1) $y'' + y = x$

(2) $y = x + \sin x + \frac{\pi}{2}\cos x$

(3) $y = x - \frac{\pi}{2}\sin x$

(4) $y'' - y = x^2$

56. $\int_0^1\int_0^1\int_0^1 \ldots\ldots\ldots \int_0^1 dx^n =$

(1) $\frac{1}{x!}\int_{1/n}^1 (x-t)^n\, dt$

(2) $\frac{1}{(n-1)!}\int_0^1 (x-t)^{n-1} dt$

(3) $\frac{1}{n(n-1)!}\int_{1/n}^1 (x-t)^{n-1} dt$

(4) $\frac{1}{(n-1)!}\int_0^1 (x-t)^{n-1} dt$

57. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions

Then–

(1) $\sup_n f_n$ is measurable

(2) $\inf_n f_n$ is measurable

(3) if $\{f_n\}_{n=0}^{\infty}$ converges pointwise to f on [a, b] then f is measurable

(4) if $\lim_{n\to\infty} f_n(x) = f(x)$ then f is measurable.

58. X_n and Y_n be two sequences of random variable Then–

(1) X_n converges to X in probability then X_n converges to X in distribution

(2) If X_n converges to be (constant) in distribution then x_n converges to be in probability

(3) x_n converges X in distribution and Y_n converges in probability to 0 then $X_n + Y_n$ converges to X in distribution

(4) If X_n converges to X in distribution and g is a continuous function $g(x_n)$ converges to g(x) in distribution.

59. A random variable x has the probability law $dF(x) = \frac{x}{b^2} e^{-x^2/2b^2} dx \quad 0 \le x < \infty$ where b is a parameter

Let L be the distance between the quartiles Then

(1) L = 0

(2) $\frac{L}{\sigma}$ is free from b

(3) $\sigma^2 = b^2\left(2 - \frac{\pi}{2}\right)$

(4) $Q_3 = b\sqrt{2}\log\sqrt{4}$

60. Let $x^2(x+1)^2 y'' + (x^2 - 1) y' + 2y = 0$

be a differential equation then–

(1) x = 0 is an ordinary point

(2) x = 0, x –1 are singular points

(3) x = 0 is irregular singular point

(4) x = –1 is regular singular point

ANSWER KEY

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Answer	2	2	2	1	3	1	3	2	2	1	3	2	3	4	3
Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Answer	1	3	4	2	4	4	3	1	2	1	4	1	1	4	3
Question	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Answer	3	3	4	3	3	1	3	1	2	2	1,2,3	1,2,3,4	1,2,3	1,3,4	1,3
Question	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Answer	1,4	1,2,4	3,4	3	2,3	1,4	1,2,4	1,2,3	1,2	1,3	2	1,2,3,4	1,2,3,4	2,3	2,3,4

HINTS AND SOLUTIONS

PART A(1-15)

1.(2) The percentage of variability over the average of that year

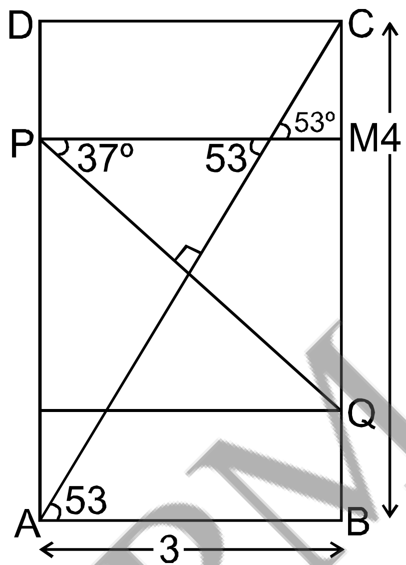
year 2000 $\left(\frac{50}{150}\times 100\right) = 33.33\%$

year 2001 $\left(\frac{75}{250}\times 100\right)=30\%$

year 2002 $\left(\frac{75}{200}\times 100\right)=37.5\%$

year 2003 $\left(\frac{50}{100}\times 100\right)=50\%$

2.(2)



Given AB = 3

BC = 4

then from using pythagoras

AC = 5 m

and $\angle CAB = \theta$

then $\tan\theta = \frac{4}{3} = \tan 53^o$ ...(i)

Then in ΔPMQ

$\sec 37^o = \frac{PQ}{PM}$ ...(ii)

Using ΔABC

$\sec 37^o = \frac{5}{4}$ ...(iii)

Using (iii) in (ii)

$\frac{5}{4} \times PM = PQ$

$PQ = \frac{5}{4} \times 3 = \frac{15}{4}$

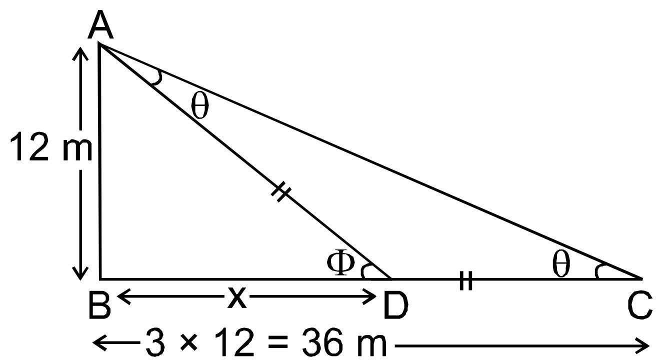
3.(2) Using allegation Formula:

Quantity of Cheaper/ Quantity of dearer= (high value-mean value)/(mean value-low value)

volume of pour Space/Volume of rice=1.5-0.80.8-0=78

So volume of pour space=1000/15×7=466.66 approximately 465.

4.(1) Figure according to question AD and CD are equal because peacock and snake has equal speed.



let ∠DAC = θ

from the fig. ∠DCA = θ

and let ∠ADB = Φ

according to Geometry

Φ = 2θ

tan 20 = tan Φ

$$\frac{2\tan\theta}{1-\tan^2\theta} = \tan\Phi$$

from the fig. $\tan\theta = \frac{12}{36} = \frac{1}{3}$

$\tan\Phi = \frac{12}{x}$

$$\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{12}{x}$$

$$\frac{\frac{2}{3}}{\frac{8}{9}} = \frac{12}{x}$$

$$\frac{2}{3} \times \frac{9}{8} = \frac{12}{x}$$

$$\frac{1}{4 \times 4} = \frac{1}{x}$$

x = 16 m

5(3) Distance of river on ground = perimeter of semi circle

= π x r

= 3.14 x 3.5

=11 cm

According to Scale = 11 x 50,000 cm

= 5,50,000 cm or 5.5 km.

6.(1) Given that the sum of squares of a nine digit number is 2. Then. The possible numbers would be

Case.I :

100000001

$1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 1^2 = 2$

Case II : 100000010

$1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 1^2 + 0^2 = 2$

Case III : 100000100

$1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 1^2 + 0^2 + 0^2 = 2$

Case IV : 100001000

$1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 1^2 + 0^2 + 0^2 + 0^2 = 2$

Case V : 100010000

$1^2 + 0^2 + 0^2 + 0^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 = 2$

Case VI : 100100000

$1^2 + 0^2 + 0^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 = 2$

Case VII : 101000000

$1^2 + 0^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 = 2$

Case VIII : 110000000

$1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 = 2$

7.(3) Given that

y = x

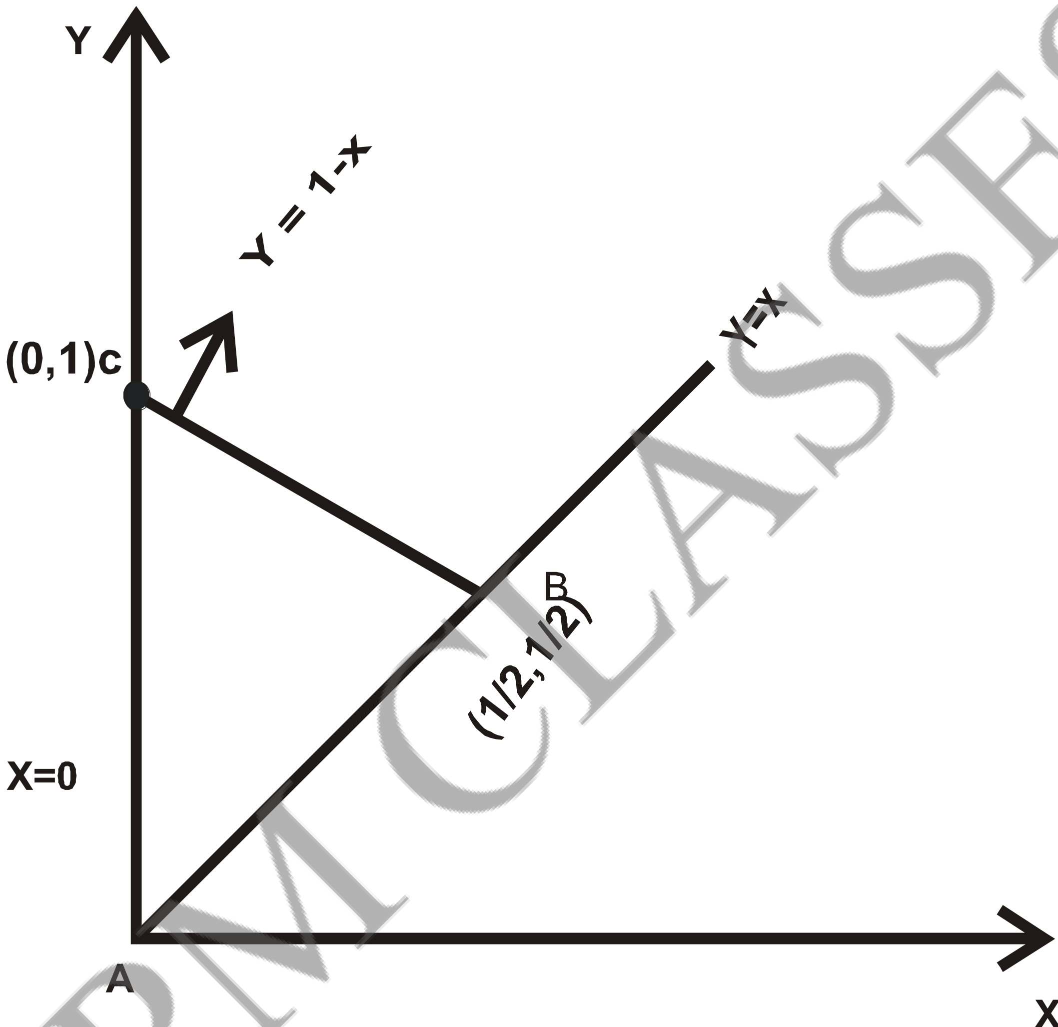
y = 1 - x and x = 0

AB = BC

& $y = x = m_1 = 1$

$y = -x + 1 = m_2 = -1$

so $m_1 m_2 = -1$, triangle is right - angled.



8.(2) The given sequence will follow the pattern 3, 4 4, 3 ....

These are the difference between two consecutive numbers of the sequence.

So.

39, 42, 46, 50 (53)

+3 +4 +4 +3

9.(2) Volume of vessel upto height 2h is equal to 2a * 2h ...(1)

Volume of vessel after removing cylinder

$= 2A * h'$ ...(2)

where h' = new height of water level.

Volume of water =Volume of vessel =Volume of solid

after removing upto height 2h Cylinder upto height cylinder h

$\Rightarrow 2A * h' = 2A * 2h - A.h$

$\Rightarrow h' = \frac{3}{2} h$

10.(1) gcd = 20 **FROM CY NTP-13 J**

lcm = 600

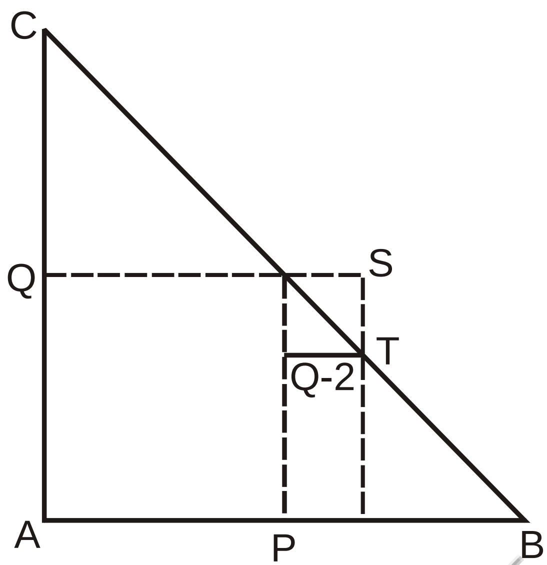
$\text{lcm} = \frac{x * y}{\text{gcd}}$

$x * y = 600 \times 20$

$x * y = 2^5 \times 3^1 \times 5^3$

x	y
$5^1 \times 2^2$	$5^2 \times 2^3 \times 3$
$5^2 \times 2^2$	$5^1 \times 2^3 \times 3$
$5^2 \times 2^3$	$5^1 \times 2^2 \times 3$
$5^2 \times 2^2 \times 3$	$5^1 \times 2^3$

11.(3)



AB = AC = 3

AQ = 2 AP

Area of $\Delta ABC = \frac{1}{2}\text{ b x h} = \frac{1}{2}(3 \times 3) = \frac{1}{2}$

Area of $\Delta QST = \frac{1}{2}(Q.2 \times Q.2) = 0.02$

Ratio = $\frac{9 \times 100}{2 \times 0.02} = 225$

12.(2) CP for the customer: 100+10=110

and 10% of 110=11 So customer sells shopkeeper at 99. SO Shopkeeper makes a profit of Rs 10+1=11.

2nd option is correct.

OR

A shopkeeper purchases a product Rs.100 and sales it making a profit 10%,then profit = 10 Rs.Again customer resells it to the same increasing a loss of 10%. Then total loss = 11 Rs

= Total profit to the shopkeeper = 1+ 10 = 11 Rs

13.(3) 500 gm of Pure coffee contains $\rightarrow$ 50 gm of chicory

100 gm of Pure coffee contains $\rightarrow$ 10 gm of chicory

Now;

5 gm is added additionally

i.e., 105 gm of coffee $\rightarrow$ 15 gm of chicory

$$\% = \frac{15}{105} x100 = \frac{100}{7} = 14.2\% \approx 14\%$$

14.(4) Raw material and Research & Development have registered increase by same percentage.

Increase in raw material from 2010 to 2011 = 6240-5200 = 1040

Percent increase = (1040/6240) x 100 = 16.6 %

Increase in Research & Development from 2010 to 2011

= 26400 - 22000 = 4400

Percent increase = (4400/26400) x 100 = 16.6 %

15.(3) The formula used in this operation is as follows :

B (+4), F(+2) H (+4), L (+2) N (+2) P (+4), T(+2) V(+2) X (+2) Z

So next one is TVXZ

PART B(16-40)

16.(1) $\bar{x}$ for population A = $\frac{101+102+...+200}{100} = \frac{\frac{100}{2}[101+200]}{100}$ = 150.5

for population B = $\frac{151+152+...+250}{100} = \frac{\frac{100}{2}[151+250]}{100}$ 200.5

$$V_A = \frac{(101-150.5)^2 + (102-150.5)^2 + ... + (200-1505)^2}{100}$$

$$= \frac{(49.5)^2 + (48.5)^2 + ... + (0.5)^2 + (0.5)^2 + (1.5)^2 + ... + (49.5)^2}{100}$$

$$V_B = \frac{(151-200.5)^2 + ... + (250-200.5)^2}{100} = \frac{(49.5)^2 + ... + (0.5)^2 + (0.5)^2 + ... + (49.5)^2}{100}$$

$$\Rightarrow \frac{V_A}{V_B} = 1$$

17. (3) Since $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from a distribution having pdf

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

We express the pdf of Y_3 in terms of f(x) and F(x) and then compute $P\left(\frac{1}{2} < Y_3\right)$.

Here $F(x) = x^2$, provided that 0 < x < 1, so that

$$g_3(y_3) = \begin{cases} \frac{4!}{2!\,1!}(y_3^2)^2(1-y_3^2)(2y_3) & 0 < y_3 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$P\left(\frac{1}{2} < Y_3\right) = \int_{1/2}^{\infty} g_3(y_3)\,dy_3$$

$$= . \int_{1/2}^{1} 24(y_3^5 - y_3^5)\,dy_3 = \frac{243}{256}$$

18. (4) (i) Since S and T are 1-1 onto, S^{-1} and T^{-1} exist.

Let $ST(x) = ST(y)$

Then $S(T(x)) = S(T(y))$

$\Rightarrow T(x) = T(y)$ as S is 1-1

$\Rightarrow x = y$ as T is 1-1 onto

$\Rightarrow$ ST is 1-1 onto

Again $ST : V \to U$, let $u \in U$ be any element then as S is in onto, $\exists w \in W$ s.t. , $S(w) = u$ and as

$T : V \to W$ is onto $\exists \nu \in V$ s.t., $T(\nu) = w$

Now $T(\nu) = w \Rightarrow S(T(\nu)) = S(w) \Rightarrow ST(\nu) = u$

or that ST is onto.

Also $(ST)(T^{-1}S^{-1}) = S(T(T^{-1}S^{-1})) = S(TT^{-1})S^{-1} = S(IS^{-1}) = SS^{-1} = I$

Similarly $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}(ST)) = T^{-1}(S^{-1}S)T = T^{-1}(IT) = T^{-1}T = I$

Showing that $(ST)^{-1} = T^{-1}S^{-1}$.

(ii) Let $\nu \in \text{Ker } T$ be any element

Then $T(\nu) = 0$

$\Rightarrow S(T(\nu)) = S(0)$

$\Rightarrow ST(\nu) = 0$

$\Rightarrow \nu \in \text{Ker } ST$ and $\text{Ker } ST = (0)$ as ST is 1-1

$\Rightarrow \nu = 0 \Rightarrow \text{Ker } T = (0) \Rightarrow$ T is 1-1 onto.

(iii) Let $u \in U$ be any element. Since $ST : V \to U$ is onto, $\exists$ some $\nu \in V$ s.t., $ST(\nu) = u$

i.e., $S(T(\nu)) = u$

Let $T(\nu) = w$ and $w \in W$ such that

$S(w) = u$

Then S is onto.

19. (2) Since $Z_1 = 3 - 4i$, $Z_2 = -4 + 3i$

and if angle between them is θ given by

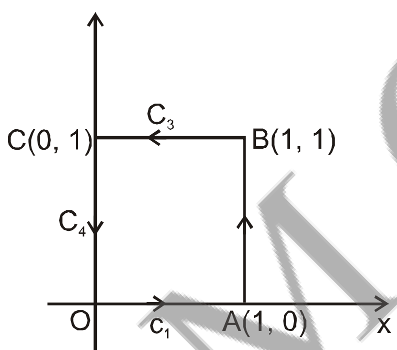
$$\cos\theta = \frac{Z_1 \cdot Z_2}{|Z_1||Z_2|}$$

$$Z_1 \cdot Z_2 = \text{Re}\{\bar{Z}_1 Z_2\} = \text{Re}\{(3 + 4i)(-4 + 3i)\} = \text{Re}\{-24 - 7i\} = -24$$

$$\cos\theta = \frac{-24}{|3-4i||-4+3i|} = \frac{-24}{25} = -0.96$$

$$\theta = \cos^{-1}(-0.96)$$

$$= \pi - \cos^{-1}(0.96)$$



20. (4)

here $I = \int_0^A x\,dz + \int_A^B x\,dz + \int_B^C x\,dz + \int_C^0 x\,dz$

along OA = x = (1 – t).0 + t.1 = t

along AB = x = (1 – t).1 + t(1 + i) = 1 + ti

along BC = x = (1 – t). (1 + i) + t.1 = 1 – t + i

along OC = x = (1 – t).i + t.0 = (1 – t)i

So $I = \int_0^1 t\,dt + \int_0^1 1.t\,dt + \int_0^1 (1-t)(-1)\,dt + 0$

$$= \frac{1}{2} + i - \frac{1}{2} + 0 \quad = i$$

21. (4) Statement A and B are true as if

$$T(z) = \frac{ax+b}{cz+d} \qquad (a, b, c, d \in c\ ad - bc \neq 0)$$

be linear transformation.

If $w = \dfrac{az+b}{cz+d}$ $\left(z \neq -\dfrac{d}{c}\right)$

it gives z = $T^{-1}(w)$ $\dfrac{dw-b}{-cw+a}$ $\left(w \neq \dfrac{a}{c}\right)$

where da – (–b) (–c) = ad – bc $\neq 0$

inverse of bilinear transformation is again a bilinear transformation.

again take $s(z) = \dfrac{a'z+b'}{c'z+d'}$ another (where a'd' – b'c' $\neq 0$)

bilinear transformation.

(To S)(z) = T(S(z)) $= \dfrac{a\left(\dfrac{a'z+b'}{c'z+d'}\right)+b}{c\left(\dfrac{a'z+b'}{c'z+d'}\right)+d} = \dfrac{(aa'+bc')z+ab'+bd'}{(ca'+dc')z+cb'+dd'}$

it is a bilinear transformation given by (ad – bc) (a'd' – b'c') $\neq 0$

(3) if a bilinear transformation fixes 1 i.e.

T(1) = 1

then $\dfrac{a+b}{c+d} = 1$

so that a = d   b = c

we conclude that T is identity map

(4) A bilinear mapping maps circles and straight lines in the z-plane into circles or lines.

22.(C) Since $f(-1) = f(1) = 2^{35}$

i.e. two real no. -1 and 1 have the same image so, the function is not one-one and let

$y = (x^2+1)^{35}$

$$x = \sqrt{y^{1/35} - 1}$$

Thus every real no. has no pre-image. So, the function is not on to.

Hence function is neither one-one nor onto.

23.(1) Let given equation is as

$f = q - 3p^2$ find $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$

$\frac{\partial f}{\partial p} = -6p, \frac{\partial f}{\partial q} = 1$

put these value in Charpit subsidiary equation

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$\Rightarrow \frac{dp}{0} = \frac{dq}{0} = ...$

$\Rightarrow$ dp = 0 dq = 0 integrate get p = a constant, q = b constant then put p and q in sz = p p dx + q dy

$\Rightarrow$ dz = a dx + b dy integrate we get

$\Rightarrow$ z = ax + by +c but $q = 3p^2$ and p = a $\Rightarrow$ $q = 3a^2$

or $z = ax + 3a^2 + c$

24.(2) Let $u_n = \frac{1}{n^P + n^{-P}} < \frac{1}{n^{|P|}}$

If $|P| > 1$, then $\sum \frac{1}{n^{|P|}}$ is Convergent, therefore by comparison test, $\sum \frac{1}{n^P + n^{-P}}$ is convergent if $|P| > 1$ and divergent if . $|P| \le 1$

25. (1) Let $2^n - 1 = p$ = prime

Let n be not a prime number

Then n is composite s.t. n = r s 1 < r, s < n

$p = 2^n - 1$

$= 2^{rs} - 1 = (2^r)^s - 1$

$= x^s - 1$ $\quad = x = 2^r > 1$ as $r > 1$

$= (x - 1)(x^{s-1} + x^{s-2} + \ldots\ldots x + 1)$

Either $x - 1 = 1$ or $x^{s-1} + x^{s-1} + x^{s-2} + \ldots\ldots\ldots x + 1 = 1$

$x - 1 = 1 \Rightarrow x = 2$ which is prime

and $x^{s-1} + \ldots\ldots + x + 1 = 1$

$x^{s-1} + \ldots\ldots\ldots x = 0$ which is natural

$\Rightarrow$ n is prime by contradiction

$\Rightarrow$ the only odd prime is n = 2

and for n =2 $\quad 2^n - 1$ = again a prime

26.(4) Ring of order 1 being the zero ring is commutative and ring of order 2 and 3 are of prime order so can prove here that rings of prime order is commutative

Let the order of ring be p(prime number)

Then <R, +> is cyclic group Let <R, + > = <a> then o(1) = o(R) = p

Let x, y $\in$ R be any elements then x = na y = ma for some integer n, m

Now xy = (na) (ma) = nma^2 = (ma) (na) = y^x

$\Rightarrow$ R is commutative.

Now we can take an example of ring of order 4

Let R be the set of 2 × 2 matrices

$\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix}\right\}$ over Z_2 with second row having zero entries Then R is a ring under matrix addition and matrix multiplication

since $\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$

we find R is non commutative and also it has order 4

27. (1) Here we have

$x_0 = 0, x_1 = 4,$

$y_0 = f(x_0) = 0,$ $y_1 = f(x_1) = 2$

$y_0^1 = f'(x_0) = 1,$ $y_1^1 = f'(x) = 0$

The Hermite's interpolating polynomial is given by

$$\phi(x) = \sum_{i=0}^{x} u_i(x) y_i + \sum_{i=0}^{1} v_i(x) y_i^1 \quad .... (1)$$

where $u_i(x) = \left[1 - 2(x - x_i)\ell_i^1(x_i)\right]\ell_i^2(x)$

and $v_i(x) = (x - x_i)\ell_i^2(x)$

we have $u_0(x) = [1 - 2(x - x_0)\ \ell_0^1\ (x_0)]\ \ell_0^2(x)$

$$= \left[1 - 2(x-0)\ell_0^1(0)\right]\ell_0^2(x)$$

Since $\ell_0(x) = \dfrac{x - x_1}{x_0 - x_1} = \dfrac{x-4}{0-4} = \dfrac{x-4}{-4}$

and $\ell_0^1(x) = -\dfrac{1}{4}$

We obtain $u_0(x) = \left[1 - 2x\left(-\dfrac{1}{4}\right)\right]\left(\dfrac{x-4}{16}\right) = \left(1 + \dfrac{x}{2}\right)\dfrac{(x-4)^2}{16}$

$$= \frac{1}{32}(x+2)(x-4)^2$$

Also we have

$$u_1(x) = \left[1 - 2(x - x_1)\ell_1^1(x)\right]\ell_1^2(x) = \left[1 - 2(x-4)\ell_1^1(x)\right]\ell_1^2(x)$$

Since $\ell_1(x) = \dfrac{x - x_0}{x_1 - x_0} = \dfrac{x-0}{4-0} = \dfrac{x}{4}$

and $\ell_1^1(x) = \dfrac{1}{4}$

We can write $u_1(x) = \left[1 - 2(x-4)\dfrac{1}{4}\right]\left(\dfrac{x^2}{16}\right) = \left(\dfrac{6-x}{2}\right)\dfrac{x^2}{16} = \dfrac{x^2(6-x)}{32}$

Similarly, we have $v_0(x) = (x - x_0)\ell_0^2(x)$

$$= (x-0) = \frac{(x-4)^2}{16} = \frac{x(x-4)^2}{16}$$

and $v_1(x) = (x - x_1)\ \ell_1^2(x) = (x-4)\dfrac{x^2}{16}$

The Hermite's polynomial is

$\phi(x) = u_0 y_0 + u_1 y_1 + v\ y_0^1 + v_1 y_1^1$

$= u_0.0 + 2u_1 + 1.v_0 + 0.v_1$

$= 2u_1 + v_1$

$= 2\left[\frac{x^2(6-x)}{32}\right] + \frac{x(x-4)^2}{16}$

$= \frac{1}{16}[6x^2 - x^3 + x^3 + 16x - 8x^2]$

$= \frac{1}{16}(16x - 2x^2) = \frac{1}{8}(8x - x^2)$

Hence the required Hermite's polynomial is $\frac{1}{8}(8x - x^2)$

28. (1) Given functional $I = \int_0^{\frac{\pi}{2}} (y'^2 - y^2 - 4y\sin^2 x)dx$

$f(x, y, y') = y'^2 - y^2 - 4y \sin^2 x$

for extremal

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-2y - 4\sin^2 x - \frac{d}{dx}(2y') = 0$

$-y - 2\sin 2x - y'' = 0$

$y'' + y = -2\sin^2 x$

$CF = C_1 \cos x + C_2 \sin x$

P.I. $= \frac{1}{D^2+1} 2\sin^2 x$

$$= \frac{1}{D^2+1}[1-\cos 2x]$$

$$= -\left(1+\frac{1}{3}\cos 2x\right)$$

so $y = c_1 \cos x + c_2 \sin x - 1 - \frac{1}{3}\cos 2x$

$y(0) = \frac{1}{3} \Rightarrow c_1 - 1 - \frac{1}{3} \Rightarrow c_1 = -\frac{4}{3}$

$y\left(\frac{\pi}{2}\right) = \frac{1}{3} \Rightarrow c_2 - 1 + \frac{1}{3} \Rightarrow c_2 = -\frac{2}{3}$

$\Rightarrow y = -\frac{4}{3}\cos x - \frac{2}{3}\sin x + 1 + \frac{1}{3}\cos 2x$

$y = (2\sin x + \cos 2x)/3$

29. (4) Let c be an Eigen value of T.

$\therefore \exists\, 0 \neq f \in V$ s.t.

Tf = cf

$\therefore$ Tf (x) = cf (x)

$\therefore \int_0^x f(t)\, dt = cf(x)$

f(x) = cf′ (x)

$y = c\frac{dy}{dx}$

$c \neq 0$ (as $c = 0 \Rightarrow y = 0 \Rightarrow f(x) = 0 \Rightarrow f = (0)$)

$\therefore \frac{dy}{y} = \frac{dx}{c}$

$\Rightarrow \log y = \frac{x}{c} + \log a \Rightarrow y = ae^{x/c}$

$\Rightarrow y(0) = a$

$\Rightarrow f(x) = y = f(0)\, e^{x/c}$

$\Rightarrow \int_0^x f(0)e^{t/c}\,dt = \int_0^x f(t)\,dt = cf(x) = cf(0)e^{x/c}$

$f(0) \neq 0$ (as $f(0) = 0 \Rightarrow a = 0 \Rightarrow y = 0 \Rightarrow f(x) = 0 \Rightarrow f = 0$)

$\therefore\ f(0)\left[ce^{t/c}\right]_0^x = cf(0)e^{x/c}$

$\therefore\ c(e^{x/c} - 1) = ce^{x/c}$

$\Rightarrow e^{x/c} - 1 = e^{x/c}$

$\Rightarrow 1 = 0$, a contradiction

$\therefore$ T has no eigen value.

30.(3) The function $\cosh\left(z + \frac{1}{z}\right)$ is analytic function everywhere at z = 0 $\phi 1$

$\therefore$ This function is analytic in the annulus $r \leq |z| \leq R$, r<R. The laurent series expansion of this function in the annulus $r \leq |z| \leq R$ is,

$$\cosh\left(z + \frac{1}{z}\right) = \sum_{-\infty}^{\infty} a_n z^n$$

where $a_n = \frac{1}{2\pi i}\int_r \cosh\left(z + \frac{1}{z}\right)\frac{dz}{z^{n+1}}$

where r is a circle , $|z| = r_1$ $(r < r_1 < R)$

Let $r_1=1$, then $|z| = 1$ and $z = e^{i\theta}$

$$\Rightarrow dz = ie^{i\theta}\, d\theta$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)\frac{ie^{i\theta}d\theta}{e^{i\theta(n+1)}}$$

$$= \frac{i}{2\pi i}\int_0^{2\pi} \cosh(2\cos\theta)e^{-in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)(\cos n\theta - i\sin n\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta \qquad \left(\therefore \int_0^{2\pi} f(\theta) = 0 \text{ if } (2\pi - \theta) = -f(\theta)\right.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\, d\theta = a_{-n}$$

Also, $\cosh\left(z + \frac{1}{z}\right)$

$$= \sum_{-\infty}^{\infty} a_n z^n = a_0 + \sum_{n=1}^{\infty} a_{-n} z^{-n} + \sum_{n=1}^{\infty} a_n z^n$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n} \right)$$

Thus $$\cosh\left(z + \frac{1}{z} \right) = a_0 + \sum_{n=1}^{\infty} \left(z^n + \frac{1}{z^n} \right) a_n$$

Where $$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta \, d\theta$$

31.(3) Let n be the number of students and p the probability for any student to need copy of a particular test book from the university library.

Mean : $\bar{X} = np = 600 \times .5 = 30$

$\sigma = \sqrt{npq} = \sqrt{600 \times .05 \times .95} = 5.34$

Let x_1 represent the number of copies of a textbook required on any day. We want x_1 such that

$P(X < z_1) > 0.9$ or $P(Z(z_1) > 0.90$

$\left(z_1 = \frac{x_1 - 30}{5.34} \right)$

or $P(0 < Z(z_1) > 0.4$

or $z_1 > 1.28$ [From normal tables]

$\therefore \frac{x_1 - \mu}{\sigma} > 1.28$ or $\frac{x_1 - 30}{5.3} > 1.28$

$x_1 - 30 > 6.784$

$x_1 > 36.784 = 37$

Hence the library should keep at least 37 copies of the book to ensure that the probability is more than 90% that none of the students reading a copy from the library has to come back disappointed.

32.(3) We have $a_0 = 1$

$$a_1 = a_0 4^{-1} = \frac{1}{4}$$

$$a_2 = a_1 4^{-2^3} = \frac{1}{4}.4^{-2^3} = \frac{1}{4^{1+2^3}}$$

$$a_3 = a_2 4^{-2^3} = \frac{1}{4^{1+2^3+3^3}}$$

$$\vdots$$

$$a_n = \frac{1}{4^{1+2^3+3^3+...+n^3}} = \frac{1}{4^{n^2(n+1)^2/4}}$$

So, the radius of convergence of power series

$$= \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n^4}} = \lim_{n\to\infty}\left(4^{n^2(n+1)^{2/4}}\right)^{1/n^4}$$

$$= \lim_{n\to\infty} 4^{\left(1+\frac{1}{n}\right)^{2/4}} = 4^{1/4} = 2^{1/2} = \sqrt{2}$$

33.(4) We have

$$g'(x) = 2f'\left(\frac{x}{2}\right).\frac{1}{2} + f'(2-x)(-1)$$

$$= f'\left(\frac{x}{2}\right) - f'(2-x)$$

Given $f''(x) < 0 \ \forall x \in (0,2)$

So f'(x) is decreases in (0,2)

Let $\frac{x}{2} > 2 - x \Rightarrow f'\left(\frac{x}{2}\right) < f'(2-x)$

Thus $f'\left(\frac{x}{2}\right) - f'(2-x) < 0$

$$\Rightarrow g'(x) < 0, \quad \forall \frac{x}{2} > 2 - x$$

$$\Rightarrow x > \frac{4}{3}$$

g decreasing in $\left(\frac{4}{3}, 2\right)$ and increasing in $\left(0, \frac{4}{3}\right)$

34. (3) Divergence Criteria If a sequence X = (x_n) of real numbers has either of the following properties, then X is divergent.

(i) X has two convergent subsequence X' = (x_{n_k}) and X" = (x_{r_k}) whose limits are not equal.

(ii) X is unbounded.

35. (3) If f is differentiable on I, the well know the implies that f continuous on I, and by the Continuous Inverse Theorem the inverse function g is continuous on J.

i.g. g' = $\frac{1}{f'og}$

36. (1) The given equation may be written as

$$g(x) = 2\lambda x \int_0^1 t\,g(t)dt - 4\lambda x^2 \int_0^1 g(t)dt \qquad ...(1)$$

or $$g(x) = 2\lambda x c_1 - 4\lambda x^2 c_2 \qquad ...(2)$$

where $$c_1 = \int_0^1 t\,g(t)\,dt \qquad ...(3)$$

and $$c_2 = \int_0^1 g(t)\,dt \qquad ...(4)$$

Using (2), (3) becomes

$$c_1 = \int_0^t t(2\lambda c_1 t - 4\lambda c_2 t^2)dt$$

$$c_1 = \left[1 - 2\lambda\int_0^1 t^2 dt\right] + 4\lambda c_2 \int_0^1 t^3\,dt = 0$$

or, $$c_1\left(1 - \frac{2\lambda}{3}\right) + \lambda c_2 = 0 \qquad ...(5)$$

Again using (2), (4) becomes

$$c_2 = \int_0^1 (2\lambda_1 ct - 4\lambda c_2 t^2)dt$$

or, $$2\lambda c_1 \int_0^1 dt - c_2\left[1 + 4\lambda\int_0^1 t^2 dt\right] = 0$$

or, $$\lambda c_1 - c_2\left(1 + \frac{4\lambda}{3}\right) = 0 \qquad ...(6)$$

For non zero solution of equations (5) and (6), we must have

$$\begin{vmatrix} 1 - \frac{2\lambda}{3} & \lambda \\ +\lambda & -\left[1 + \frac{4\lambda}{3}\right] \end{vmatrix} = 0$$

or, $-\left(1-\frac{2\lambda}{3}\right)\left(1+\frac{4\lambda}{3}\right)-\lambda^2=0$

or, $-1-\frac{2\lambda}{3}+\frac{8\lambda^2}{9}-\lambda^2=0$

or, $\lambda^2+6\lambda+9=0$

37. (3) Here $f(x) = x,\ \lambda = 1,\ k(x, t) = 1$ ...(1)

We know that $k_1(x, t) = k(x, t) = 1$ ...(2)

The n^{th} iterated kernal is given by

$$k_n(x, t) = \int_0^{1/2} k(x,z)k_{n-1}(z, t)\,dz \quad ...(3)$$

On putting n = 2 sin (3), we have

$$k_2(x, t) = \int_0^{1/2} k(x,z)\,k_1(z, t)\,dz$$

$$= \int_0^{1/2} 1\,dz \quad \text{[using (2)]}$$

or $k_2(x, t) = [z]_0^{1/2} = \frac{1}{2}$ ...(4)

Again putting n = 3 in (3), we have

$$k_3(x, t) = \int_0^{1/2} k(x,z)k_2(z,t)\,dz$$

$$= \int_0^{1/2} \frac{1}{2}\,dz \quad \text{[using (2) and (4)]}$$

$$k_3(x, t) = \left(\frac{1}{2}\right)^2 \quad ...(5)$$

and so on repeating the above process, we have in general

$$k_n(x, t) = \left(\frac{1}{2}\right)^{n-1} \qquad \text{...(6)}$$

The resolvent kernal R(x, t; λ) is given by

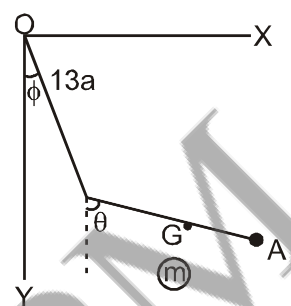
$$R(x, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \qquad \text{[using (6)]}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + ...$$

$$R(x, t; \lambda) = \frac{1}{1-1/2} = 2$$

38. (1) We have



$x_G = 13\,a \sin\phi + 4a \sin\theta$

and $y_G = 13a \cos\phi + 4a \cos\theta$

$$\therefore \quad \dot{x}_G^2 + \dot{y}_G^2 = 169\,\dot{a}^2\,\dot{\phi}^2 + 16a^2\,\dot{\theta}^2$$

$$+ 104\,a^2 \cos(\theta - \phi)$$

Thus, $T = \frac{1}{2} m\,[k^2\dot{\theta}^2 + (\dot{x}_G^2 + \dot{y}_G^2)] = \frac{1}{2} ma^2 \left[\frac{64}{3}\dot{\theta}^2 + 169\dot{\phi}^2 + 104\,\dot{\theta}\,\dot{\phi}\right]$ and the work function

$W = mg\,(13a\cos\phi + 4a\cos\theta)$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} \Rightarrow 61\,\ddot{\theta} + 39\,\ddot{\phi} = -\frac{3g}{a}\theta$$

39.(2) We have $\varepsilon_{n+1} = \varepsilon_n - \dfrac{2f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$ where a, ε_n, ε_{n+1} have their usual meanings, Expanding in powers of ε_n and using f(1) = 0, f′(1) = 0, since x = a is a double root x = x_n, we get

$$\varepsilon_{n+1} = \varepsilon_n - \frac{2\left[\frac{\varepsilon_n^2}{2!}f''(a)+\ldots\right]}{\left[\varepsilon_n f^n(a)+\frac{\varepsilon_n^2}{2!}f''(a)+\ldots\right]}$$

$$= \varepsilon_n - \frac{2.\varepsilon_n^2\frac{1}{2!}\left[f''(a)+\frac{1}{3!}\varepsilon_n f'''(a)+\ldots\right]}{\varepsilon_n\left[f''(a)+\frac{\varepsilon_n}{2!}f'''(a)+\ldots\right]}$$

$$= \varepsilon_n - \frac{2\varepsilon_n\left[\frac{1}{2!}f''(a)+\frac{1}{3!}\varepsilon_n f'''(a)\right]}{\left[f''(a)+\frac{\varepsilon_n}{2!}f'''(a)\right]}$$

$$ \varepsilon_n - \frac{2\varepsilon_n\left[\frac{1}{2!}f''(a)+\frac{1}{3!}\varepsilon_n f'''(a)\right]}{\left[f''(a)+\frac{\varepsilon_n}{2!}f'''(a)\right]}$$

$$ \frac{1}{6}\varepsilon_n^2.\frac{f'''(a)}{\left[f''(a)+\frac{\varepsilon_n f'''(a)}{2!}\right]}$$

$$\varepsilon_{n+1} \approx \frac{1}{6}\varepsilon_n^2 . \frac{f'''(a)}{[f''(a)]}$$

which shows that $\varepsilon_{n+1} \propto \varepsilon_n^2$ and hence the convergence is quadratic.

40. (2) Now $T(\in_1) = T(1, 0)$

$= (1, 0)$

$= a(1, i) + b(-i, 2)$

$\Rightarrow$ $a - bi = 1$ where $a, b \in C$

$ai + 2b = 0$

$\Rightarrow$ $a = 2, b = -i$

$\Rightarrow$ $T(\in_1) = 2\alpha_1 - i\alpha_2$

Also $T(\in_2) = T(0, 1) = (0, 0) = 0\alpha_1 + 0\alpha_2$

$$\therefore \quad [T]_{\beta\,\beta'} = \begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}$$

PART C(41-60)

41.(1,2,3) Let R_∞ be the extended set of real numbers (i.e., the set of real numbers including $-\infty$ and $+\infty$).

he functions d defined by

$d(x,y) = |f(x) - f(y)|, \ \forall\, x, y \in \mathbf{R}_\infty$

where f(x) is given by

$$f(x) = \begin{cases} \dfrac{x}{1+|x|}, & \text{when } -\infty < x < \infty \\ 1, & \text{when } \quad x = \infty \\ -1, & \text{when } \quad x = -\infty \end{cases}$$

Show that ($\mathbf{R}_\infty$, d) is a bounded metric space.

For the triangle inequality

$$d(x,y) = \left|\frac{x}{1+|x|} - \frac{x}{1+|y|}\right|$$

$$= \left|\frac{x}{1+|x|} - \frac{z}{1+|z|} + \frac{z}{1+|z|} - \frac{y}{1+|y|}\right|$$

$$\le \left|\frac{x}{1+|x|} - \frac{z}{1+|z|}\right| + \left|\frac{z}{1+|z|} - \frac{y}{1+|y|}\right|$$

= d(x, z) + d(z, y) , x, y, z $\in \mathbf{R}$

If x = ∞, y = –∞, then

$$d(x, y) = |1 - (-1)| \le \left|1 - \frac{z}{1+|z|}\right| + \left|\frac{z}{1+|z|} - (-1)\right|$$

Similarly, when x = –∞, y + ∞, then triangle inequality holds.

Hence ($\mathbf{R}_\infty$, d) is metric space.

Moreover, if x and y are two elements of $\mathbf{R}_\infty$, then

$$-1 \le f(x) \le 1, \quad \text{and} \quad -1 \le f(y) \le 1,$$

$\therefore$ d(x, y) = |f(x) – f(y)| x, y $\in R_\infty$

Hence (R_∞ d) is a bounded metric space.

42.(1,2,3,4) Let $T(z)=\dfrac{az+b}{cz+d}$, $ad-bc\neq c$, be any bilinear transformation.

If c = 0, then $T(z)=\dfrac{a}{d}z+\dfrac{b}{d}=Az+B$

When $A=\dfrac{a}{d}, B=\dfrac{b}{d}$

Clearly Az+B, being linear, maps circles and lines into circles and lines (a line is a circle with infinite radius)

If $c\neq 0$, then

$$T(z)=\frac{a(z+d/c)}{c(z+d/c)}+\frac{b}{cz+d}-\frac{ad}{c(cz+d)}$$

$$=\frac{a}{c}+\frac{bc-ad}{c(cz+d)}$$

$$=\frac{a}{c}+\frac{bc-ad}{c^2}.\frac{1}{z+d/c}$$

Putting $z_1=z+\dfrac{d}{c}, z_2=\dfrac{1}{z_1}$

$$z_3=\frac{bc-ad}{c^2}.z_2$$

We get $T(z)=\dfrac{a}{c}+z_3$

Which is of the form

$$\omega_1 = z + \alpha,\ \omega_2 = \frac{1}{z},\ \omega_3 = \beta z$$

Which shows that every bilinear transformation is the resultant of bilinear transformation with simple geometric imports. Thus, a bilinear transformation maps circle and lines into circle and lines.

43.(1,2,3) We have

$$\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right\} = \Delta^2\left(\frac{5x+12}{(x+2)(x+3)}\right)$$

$$= \Delta^2\left(\frac{2}{x+2}+\frac{3}{x+3}\right)$$

$$= \Delta\left(\Delta\left(\frac{2}{x+2}\right)+\Delta\left(\frac{3}{x+3}\right)\right)$$

$$= \Delta\left[2\Delta\left(\frac{1}{x+2}\right)+3\Delta\left(\frac{1}{x+3}\right)\right]$$

$$= \Delta\left[2\left(\frac{1}{x+3}-\frac{1}{x+2}\right)+3\left(\frac{1}{x+4}-\frac{1}{x+3}\right)\right]$$

$$= 2\Delta\left[\left(\frac{1}{x+3}-\frac{1}{(x+2)}\right)+3\Delta\left(\frac{1}{x+4}-\frac{1}{x+3}\right)\right]$$

$$= -2\Delta\left(\frac{1}{(x+3)(x+2)}\right) - 3\Delta\left(\frac{1}{(x+4)(x+3)}\right)$$

$$= -2\left[\frac{1}{(x+4)(x+3)} - \frac{1}{(x+3)(x+2)}\right]$$

$$-3\left[\frac{1}{(x+5)(x+4)} - \frac{1}{(x+4)(x+3)}\right]$$

$$= -2\frac{(x+2-x-4)}{(x+2)(x+3)(x+4)} - 3\frac{(x+3-x-5)}{(x+3)(x+4)(x+5)}$$

$$= -2\frac{(-2)}{(x+2)(x+3)(x+4)} - 3\frac{(-2)}{(x+3)(x+4)(x+5)}$$

$$= \frac{4(x+5)+6(x+2)}{(x+2)(x+3)(x+4)(x+5)}$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

44.(1,3,4) Given $|a_1|<|a_2|$ and $\frac{a_2}{a_1}$ so let $a_1 = 1$, $a_2 = -2$

Now $|a_2|<|a_3|$ and $\frac{a_3}{a_2}$ so let $a_3 = 4 = 2^2$

$|a_3|<|a_4|$ and $\frac{a_4}{a_3}$ so let $a_4 = -8 = -2^3$

$|a_4|<|a_5|$ and $\frac{a_5}{a_4}$ so let $a_5 = 16 = 2^4$

So $a_n = (-1)^{n-1}\, 2^{n-1}$

$$\sum \frac{1}{a_n} = \sum \frac{(-1)^{n-1}}{2^{n-1}}$$ be a convergent series and converge absolutely

$\Rightarrow$ Option (a) is incorrect and also (d) is incorrect

Here $\sum a_n = \sum 2^{n-1}$ be a geometric series with common ratio 2>1, So it is divergent.

Thus option (c) cannot be true

we have $|a_n| < |a_{n+1}|, \forall n \in \mathbb{N}$

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \ \forall n \in \mathbb{N}$$

$$\Rightarrow \left|\frac{\frac{1}{a_n}}{\frac{1}{a_{n+1}}}\right| > 1 \ \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{1}{a_n}}{\frac{1}{a_{n+1}}}\right| > 1 \Rightarrow \sum \frac{1}{a_n}$$ converge absolutely.

45.(1,3) $T : V \to V$

$T^2 : V \to V$

Rank T^2 = dim V – dim Ket T^2

$\Rightarrow$ dim Ker T = dim Ker T^2

$\Rightarrow$ nullity T = nullity T^2

If we claim Ker T = Ker T^2

x $\in$ Ker T $\Rightarrow$ T(1) = 0

$T^2(x) = 0$

T(0) = 0

$\Rightarrow$ x $\in$ Ket T^2 $\Rightarrow$ Ker T $\subseteq$ Ket T^2

$\Rightarrow$ Ket T = Ket T^2 (as they have same dimension)

now x $\in$ Range $T \cap$ Ker T

$\Rightarrow$ X $\in$ Range T and x $\in$ Ket T

$\Rightarrow$ T(x) = 0 , x = T(y) for some y $\in$ V

T(Y) = 0

$T^2(y) = 0$

y $\in$ Ket T^2 = Ket T

$\Rightarrow$ T(y) = 0

x = 0

Ket $T \cap$ Range T = {0}

46.(1,4) Let X = (x_1, x_2), Y = (y_1, y_2), $\alpha, \beta \in$

(a) $T(\alpha X + \beta Y) = T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$

$= T(\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1) = \alpha(x_2, x_1) + \beta(y_2, y_1) = \alpha T(X) + \beta T(Y)$

$\Rightarrow T$ is linear.

(b) T(X+Y) = T(x_1+y_1, x_2+y_2)

$= (\text{Sin}(x_1+y_1), x_2+y_2)$ T(X)+T(Y)

$\Rightarrow$ T is not linear

(c) T(X + Y)

$= T(x_1+y_1, x_2+y_2)$

$= ((x_1+y_1)^2, x_2+y_2)$ T(X)+T(Y)

$\Rightarrow$ T not linear

(d) $T(x_1, y_2)$

$= (x_1 - x_2, 0)$ is linear.

47.(1,2,4) Here $u_1(x) = x$,

$u_2(x) = x^{1/3} - x$,

$u_3(x) = x^{1/5} - x^{1/3}$,

… … … …

$u_n(x) = x^{1/2(2n-1)} - x^{1/(2n-3)}$.

Hence $f_n(x) = x^{1/(2n-1)}$.

$\therefore$ f(0) = 0

and f(0) = 1 for all other values of x.

Hence f is discontinuous at x = 0 and consequently zero is a point of non-uniform convergence of the series.

Now for $0 \le x \le c < \infty$, we have

$$\int_0^c f(x)\,dx = \int_0^c dx = c$$

and $\int_0^c f_n(x)\,dx = \int_0^c x^{1/(2n-1)}dx$

$$= \frac{2n-1}{2n} c^{2n/(2n-1)} \to c \text{ as } n \to \infty.$$

Hence the series is term by term integrable in the above interval although 0 is a point of non-uniform convergence of the series.

48.(3,4) We have $\frac{g(z+h)-g(z)}{h} = \frac{\overline{f}\left(\overline{z+h}-f(\overline{z})\right)}{h} = \frac{\overline{f(\overline{z}+\overline{h})-f(\overline{z})}}{h}$

$$= \overline{\left(\frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}\right)}$$

$$\lim_{h\to 0} g\frac{(z+h)-g(z)}{h} = \lim_{h\to 0} \overline{\frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}}$$

$$= \overline{\left(\lim_{h\to 0} \frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}\right)} = \overline{\frac{df(\overline{z})}{dz}}$$

Thus g(z) has a derivative at z and the derivative is equal to the complex conjugate of the derivative of f at $\overline{z}$

Since this hold for all $z \in D^*$

Thus g is analytic in $D^* = \{z : \overline{z} \in d\}$

if f'(z) = 0 in domain D

Since f(z) = u(x, y) + iv(x, y)

now f'(z) = 0 $\Rightarrow$ $u_x + iv_x = 0$

$\Rightarrow$ $u_x = 0 = v_x$ $\quad \forall (x,y) \in D$

By Cauchy riemann equations $u_y = v_y = 0$ $\forall (x,y) \in D$

so the gradient vector $\nabla_u = (u_x, u_y) = (0, 0)$ is zero

$\Rightarrow$ directional derivative of u (x, y) is zero in all directions

Hence u (x, y) is constant along a line segment joining two points.

Thus f(z) is free from z

49.(3) By definition we know that

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)\delta z_j$$

Now

$$\left|\sum_{j=1}^{n} f(\xi_j)\delta z_j\right| \le \sum_{j=1}^{n} |f(\xi_j)|\,|\delta z_j|$$

$$\le \sum_{j=1}^{n} M|\delta z_j|, \qquad \because |f(z)| \le M$$

for all points z on C (given)

i.e. $\le M\sum_{j=1}^{n}|\delta z_j|$ .... (ii)

Now $\sum_{j=1}^{n}|\delta z_j|$ represents the sum of all the chord lengths joining points zj – 1 and zj, where j = 1, 2, 3, ., n and so this sum cannot be greater than (i.e. is equal to or less than) the length ℓ of the curve C.

$\therefore$ From (ii) we have $\left|\sum_{j=1}^{n} f(\xi_j)\delta z_j\right| \le M\ell$ ... (iii)

Taking the limit $(n\to\infty)$ of both sides of (iii), from (i) we get

$$\left|\int_C f(\xi)\delta z\right| \le \ell m$$

Hence proved.

50.(2,3) Since $x^2 + 2x + 2$ is irreducible over $Z_5[x]$

$\Rightarrow \dfrac{Z_5[z]}{\langle x^2 - 2x + 2\rangle}$ is a field

$\Rightarrow$ no. of elements of field is 5^2 = 25 and since 1 and 2 are only two divisors of 2

$\Rightarrow$ no. of subfields of $\dfrac{Z_5[x]}{\langle x^2 + 2x + 2\rangle} = 2$

But $x^2 - 2x + 15$ is reducible over $Z_5[x]$

$\Rightarrow \dfrac{Z_5[x]}{\langle x^2 - 2x + 5\rangle}$ is not a field

Hence not isomorphic to $\dfrac{Z_5[x]}{\langle x^2 + 2x + 2\rangle}$

51.(1,4) We know if $P(x) \in F[x]$ then $\dfrac{F[x]}{< P(x) >}$ is a field iff P(x) is irreducible polynomial over F.

Now if $P(x) = x^2+1$, then if is irreducible over □ .

$P(x) = x^2-1 = (x-1)(x+1)$ if is reducible over □ .

$P(x) = x^3-x^2+x-1 = (x-1)(x^2+1)$ it is reducible over □ .

$P(x) = x^2+x+1$, if is irreducible over □ .

$\Rightarrow$ (a) and (d) option will be correct.

52.(1,2,4) Here we are given $P_3(x) = x^3 - 5x^2 + 17x - 3$, ... (i)

where $0 \le x \le 4$

First of all, we change the interval from [0, 4] to [–1, 1] by using the transformation

Then $P_3(z) = [2(z + 1)]3 - 5[4(z + 1)^2] + 17[2(z + 1)] - 3$

$= 8(z^3 + 3z^2 + 3z + 1) – 20(z^2 + 2z + 1) + 34(z + 1) – 3$

$= 8z^3 + 4z_2 + 18z + 19, \quad -1 \le z \le 1$

$= 8\left[\frac{1}{4}(3T_1 + T_3)\right] + 4\left[(T_0 + T_2)\right] + 18[T_1] + 19[T_0],$ expressing each power of z in terms o Chebyshev polynomials

$= 21\ T_0 + 24T_1 + 2T_2 + 2T_3$, where $-1 \le z \le 1$

Now truncating this polynomial at T_2, we have

$= \max_{-1\le z\le 1} |P_3(z) - (21T_0 + 24T_1 + 2T_2)|$

$= \max_{-1\le z\le 1} |2T_3|,$ where $T_3(z) = 4z^3 – 3z$

$= 2 = \min_{-1\le z\le 1} |P_3(z) – (21T_0 + 24T_1 + 2T_2)|$

Hence the required approximation is

$P_2(z) = 21\ To\ (z) = 24\ T_1(z) + 2\ T_2(z)$

$= 21(1) + 24(z) + 2(2z^2 – 1)$ ...

or $P_2(z) = 4z^2 + 24z + 19,$ ... (iii)

for which the maximum absolute error $\delta = 2$ is as small as possible.

Now from (ii) we have $z = (x – 2)/2$

Substituting this value in (iii) we have

$P_2(x) = 4[(x – 2)^2/4] + 24[(x – 2)/2] + 19$

$= (x^2 – 4x + 4) + 12(x – 2) + 19$

or $P_2(x) = x^2 + 8x – 1$

Also we find that $\left|P_3(x)-P_2(x)\right|$

$$=\left|\left(x^3-5x^2+17x-3\right)-\left(x^2+8x-1\right)\right|$$

$$=\left|x^3-6x^2 9x-2\right|=\delta=2$$

for x = 0, 1, 3 and 4

53.(1,2,3) First we consider vectors $u_1\ u_2 ... u_n$ is $\mathbb{R}^n$ form an orthonormal set if they are unit vectors and are orthogonal to each other where the dot product in $\mathbb{R}^n$ is defined by $(a_1\ a_2......a_n).(b_1 b_2......b_n) = a_1\bar{b}_1+a_2\bar{b}+...a_n\bar{p}_n$

Suppose A is unitary and $R_1, R_2 .... R_n$ are its rows then $\bar{R}_1^T, \bar{R}_2^T, ...\bar{R}_n^T$ are the columns of AH Let $AA^H + [c_{ij}]$ by matrix multiplication c $c_{ij}=R_i\bar{R}^T{}_J=R_i R_j$

Since A is unitary we have AAH = I

A is orthogonal also multiplying A and A and setting each entry C_{ij} equal to the corresponding entry in I yields the following n^2 equations

$R_i . R_1 = 1$ $\qquad$ $R_2 . R_1 = 1 ....R_n . R_n = 1$

and $R_i . R_j = 0$ $\qquad$ for $i \neq j$

Thus the rows of A are unit vectors and are orthogonal to each other

Hence they form an orthonormal set of vector

The condition $A^T A = I$ show that the columns of A also form an orthonormal set of vectors

If we take vectors in $\mathbb{R}^n$ then only orthonormal vectors can follows the above process they may not unitary.

54.(1, 2) The given differential equation is

$$\left[\frac{d}{dx}\left(x\frac{d}{dx}\right)-\frac{n^2}{x}\right]u=0 \quad \text{... (1)}$$

with the boundary conditions u(0) = 0 and u(1) = 0 .... (2, 3)

Comparing the equation (1) with the operator

$$\left[\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right],$$

we have p(x) = x p(ξ) = ξ .... (4)

$$\Rightarrow \quad x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-n^2u=0 \quad \text{.... (5)}$$

The general solution of the equation (5) is given by

$u(x) = Ax^n + Bx^{-n}$ .... (6)

The functions $u_1(x) = x_n$ and $u_2(x) = \left(\frac{1}{x^n}-x^n\right)$ are, respectively, linearly independent solutions of the equation (5) the satisfy the conditions u (0) = 0 and u(1) = 0 The Wronskian of $u_1(x)$ and $u_2(x)$ is given by

$$W[u_1(x), u_2(x)] = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} x^n & \frac{1}{x^n}-xn \\ nx^{n-1} & -\frac{n}{x^{n+1}}-nx^{n-1} \end{vmatrix} = -\frac{2n}{x} \neq 0,$$

which shows that $u_1(x)$ and $u_2(x)$ are two linearly independent solutions.

We know than the Green's function G(x, ξ) is given by

$$G(x,\xi) = \begin{cases} -\frac{1}{C}u_1(x)u_2(\xi), & x<\xi \\ -\frac{1}{C}u1(\xi)\,u_2(x), & x>\xi \end{cases}$$

where C is given by the Abel's formula

$u_1(x)u_2'(x) - u'_1(x)\, u_2(\xi) = \dfrac{C}{p(\xi)}$ .... (7)

where $u_1(\xi) = \xi_n$ and $u_2(\xi) = \dfrac{1}{\xi^n} - \xi^n$

Substituting the value of $u_1(\xi)$, $u_2(\xi)$, we have

The Green's function $G(x, \xi)$ becomes

$$G(x,\xi) = \begin{cases} \dfrac{x^n}{2n\xi^n}\left(1-\xi^{2n}\right), x < \xi \\ \dfrac{\xi^n}{2n\,x^n}\left(1-x^{2n}\right), x > \xi \end{cases}$$

55.(1,3) Let $F = y'^2 - y^2 + 2xy$

$$\frac{\partial F}{\partial y} = -2y + 2x$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

Euler's equation $\dfrac{\partial F}{\partial y'} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

becomes $-2y + 2x - 2y'' = 0$

$$y'' + y = x$$

$$(D^2 + 1)y = x$$

C.F. $= c_1\cos x + c_2\sin x$

PI $= \dfrac{1}{D^2+1}x = x$

Thus $y = c_1 \cos x + c_2 \sin x + x$

using boundary conditions

$x = 0 \quad y = 0 \quad c_1 = 0$

$x = \frac{\pi}{2} \quad y = 0 \quad \Rightarrow c_2 = -\frac{\pi}{2}$

$y = x - \frac{\pi}{2}\sin x$

56.(2) Let $I_n = \int_a^x (x-t)^{n-1} f(t)dt$ ...(1)

where n is a positive integer and a is a constant.

Differentiating both sides with respect to x,

we get

$\frac{dI_n}{dx} = \int_a^x \frac{\partial}{\partial x}(x-t)^{n-1} f(t)\,dt + [(x-t)^{n-1} f(t)]_{t=x} \cdot 1 - [(x-t)^{n-1} f(t)]_{t=a} \cdot 0$

$= \int_a^x (n-1)(x-t)^{n-2} f(t)\,dt$

$= (n-1)\, I_{n-1}(x)$ ...(2)

Differentiating (2) with respect to x

$\frac{d^2 I_n}{dx^2} = (n-1)\frac{d}{dx}[I_{n-1}(x)]$

$= (n-1)(n-2) I_{n-2}$, [using (1)]

Proceeding in this way, we get,

$\frac{d^2 I_n}{dx^{n-1}} = (n-1)(n-2)\ldots 1.\ \dot{I}_1(x)$

$= (n-1)!\, I_1(x)$

Now taking n = 1 in (1), we get

$I_1 = \int_a^x f(t)dt = \int_a^x f(x_1)dx_1$ ...(3)

Putting x = a in (1), we obtain

$I_n(1) = 0$ for all n

Taking n = 2 in (2), we get

$\frac{dI_2}{dx} = I_1(x)$

$\therefore\ I_2 = \int_a^x I_1(x_2)dx_2$ [$\because I_2(1) = 0$]

$= \int_a^x \int_a^{x_2} f(x_1)dx_1 dx_2$ ....(4)

Putting n = 3 in (2), we have

$\frac{dI_3}{dx} = 2I_2(x)$

$\therefore\ I_3 = 2\int_a^x I_2(x)dx$ [$\because I_3(1) = 0$]

$= 2\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1\, dx_2\, dx_3$

Proceeding in this way, we get

$I_n = (n-1)! \int_a^x \int_a^{x_n} ... \int_a^{x_2} F(x_1)dx_1\, dx_2 .... dx_n$

or $\int_a^x \int_a^{x_n} .... \int_a^{x_2} f(x_1)dx_1 dx_2 .... dx_n$

$= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt.$

57.(1,2,3.4) Let h(x) = Sup $\{f_1(x), f_2(x) ..... f_n(x)\}$

then $\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_1(x) > \alpha\}$ Since $f_i(x)$ are measurable the union of measurable functions is also measurable

Let $g(x) = \operatorname{Sup}_n f_n(x)$

then $\{x : g(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$

Since $f_i(x)$ are measurable then union of measurable function is also measurable so g is measurable.

again. Let $g_1(x) = \inf \{f_i(x)\}$

then $\{x : g_1(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) < \alpha\}$

Since $f_i(x)$ are measurable the union of measurable function is also measurable so g_1 is measurable

Let $h_1(x) = \inf_n f_n(x)$

then $\{x : h_1(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) < \alpha\}$

Since $f_i(x)$ are measurable the union of measurable function is also measurable so h_1 is measurable.

For $n \in I$ let

$g_n(x) = \text{l.u.b.} \{f_n(x), f_{n+1}(x), f_{n+2}(x) \ldots\}$ $(a \le x \le b)$.

Then, solution of each gn is a measurable function, Moreover,

$f^*(x) = \lim_{n\to\infty}$ $(a \le x \le b)$.

Also, for any $x \in [a, b]$,

$$g_1(x) \ge g_2(x) \ge g_3(x) \ge \ldots .$$

Hence, if $s \in R$,

$$\{x \mid f^*(x) < s\} = \bigcup_{n=1}^{\infty} \{x \mid g_n(x) < s\}.$$

It follows that f* is measurable.

That f_* is measurable may be proved similarly. Finally, if $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f, then f $= f^* = f_*$ and so f is measurable.

Let E be the set of x in [a, b] at which the statement.

$$\lim_{n\to\infty} f_n(x) = (x)$$

does not hold. Then, by hypothesis, E has measure zero. Define the functions $g_n (n \in I)$ and g as follows:

$g_n(x) = f_n(x) \quad (x \notin E);$ $\qquad g(x) = f(x) \quad (x \notin E)$

$g_n(x) = 0 \quad (x \in E)$ $\qquad g(x) = 0 \quad (x \in E).$

Then each g_n is measurable, Now, if $x \in E$, then

$$\lim_{n\to\infty} g_n(x) = 0 = g(x).$$

Also, if $x \notin E$, then

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x) = g(x).$$

Hence $\{g_n\}_{n=1}^{\infty}$ converges pointwise (everywhere) to g on [a, b]. Since each g_n is measurable, thus g is measurable.

58.(1,2,3,4) Let x be a point of continuity of $F_X(x)$. Let $x > 0$. We have,

$FX_n(x) = P[X_n \le x]$

$= P|\{X_n \le x\} \cap \{|X_n - X| < \varepsilon\}| + P[\{X_n \le x\} \cap \{|X_n - X| \ge \varepsilon\}]$

$\le P[X \le x + \varepsilon] + P[|X_n - X| \ge \varepsilon].$

Based on the inequality and the fact that $X_n \xrightarrow{P} X,$ we see that

$$\overline{\lim_{n\to\infty}} F_{X_n}(x) \le F_X(x+\varepsilon). \quad .... (1)$$

To get a lower bounded, we proceed similarly with the complement to show that

$P[X_n > x] \le P[X \ge x - \varepsilon] + P[|X_n - X| \ge \varepsilon].$

Hence,

$$\underline{\lim_{n\to\infty}} F_{Xn}(x) \ge F_X(x-\varepsilon). \quad .... (2)$$

Using a relationship between $\overline{\lim}$ and $\underline{\lim}$, it follows from (1) and (2) that

$$F_X(x-\varepsilon) \le \underline{\lim_{n\to\infty}} F_{Xn}(x) \le \overline{\lim_{n\to\infty}} F_{Xn}(x) \le F_X(x+\in).$$

Letting $\in \to 0$ gives us the desired result.

Let $\in$ > 0 be given. Then,

$\lim_{n\to\infty} P[|X_n - b| \le \in] = -\lim_{n\to\infty} F_{X_n}(b+\in) - \lim_{n\to\infty} F_{Xn}(b-\in -0) = 1-0 = 1,$

which is the desired result.

Suppose X_n converges to X in distribution and Y_n converges in probability to 0. Then $X_n + Y_n$ converges to X in distribution.

Suppose X_n converges to X in distribution and g is a continuous function on the support of X. Then $g(X_n)$ converges to g(X) in distribution.

59.(2, 3) If Q_1 and Q_3 are the first and third quartiles respectively, we have

$\int_0^{Q_1} f(x)dx = \frac{1}{4} \Rightarrow \frac{1}{b^2}\int_0^{Q_1} xe^{-x^2/2b^2}\,dx = \frac{1}{4}.$ Put $y = \frac{x^2}{2b^2}$ so that $dy = \frac{x}{b^2}dx.$

$\therefore \int_0^{Q_1^2/2b^2} e^{-y}\,dy = \frac{1}{4} \Rightarrow \left|\frac{e^{-y}}{-1}\right|_0^{Q_1^2/2b^2} = \frac{1}{4} \Rightarrow 1 - e^{-Q_1^2/2b^2} = \frac{1}{4} \Rightarrow e - Q_1^2/2b^2 = \frac{3}{4}$

Thus $-\frac{Q_1^2}{2b^2} = \log\left(\frac{3}{4}\right) \frac{Q_1^2}{2b^2} = \log\left(\frac{4}{3}\right) \Rightarrow Q_1 = b\sqrt{2}\,\sqrt{\log(4/3)}$

Again we have $\int_0^{Q_3} f(x)\,dx = \frac{3}{4}$ which, on proceeding similarly, will give

$1 - e^{Q_3^2/2b^2} = \frac{3}{4} \Rightarrow e^{-Q_3^2/2b^2} = \frac{1}{4} \Rightarrow Q_3 = b\sqrt{2}\,\sqrt{\log 4}$

The distance between the quartiles is : $Q_3 - Q_1 = b\sqrt{2}\left\{\sqrt{\log 4} - \sqrt{\log(4/3)}\right\}$

$\mu_1' = \int_0^\infty x f(x)\,dx = \int_0^\infty x\frac{x}{b^2}e^{-x^2/2b^2}dx = \int_0^\infty b\sqrt{2}\,y^{1/2}\,d^{-y}dy$

$= b\sqrt{2}\int_0^\infty e^{-y}\,y^{\frac{3}{2}-1}dy = b\sqrt{2}\,\Gamma(3/2) = b\sqrt{2}\frac{1}{2}\Gamma(1/2) = b\sqrt{2}\frac{\sqrt{\pi}}{2} = b\sqrt{\pi/2}$

$\mu_2' = \int_0^\infty x^2 f(x)dx = \int_0^\infty x^2\frac{x}{b^2}e^{-x^2/2b^2}\,dx$

$= 2b^2\int_0^\infty y\,e^{-y}\,dy, \left(y = x^2/2b^2\right) = 2b^2\Gamma(2) = 2b^2.1! = 2b^2$

$\therefore \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = 2b^2 - b^2\cdot\frac{\pi}{2} = b^2\left(2 - \frac{\pi}{2}\right) \Rightarrow \sigma = b\sqrt{\{2-(\pi/2)\}}$

Hence $\frac{Q_1 - Q_1}{\sigma} = \frac{\sqrt{2}\left[\sqrt{\log 4} - \sqrt{\log(4/3)}\right]}{\sqrt{\{2-(\pi/2)\}}},$ which is independent of parameter 'b'.

60.(2,3,4) On changing given differential equation into standard form.

$$\frac{d^2y}{dx^2}+\frac{x-1}{x^2(x+1)}\frac{dy}{dx}+\frac{2}{x^2(x+1)^2}y=0$$

$$P(x)=\frac{x-1}{x^2(x+1)} \qquad Q(x)=\frac{2}{x^2(x+1)^2}$$

P(x) and Q(x) are undefined at x = 0 and x = –1 they are not analytic at x = 0 and –1

⇒ x = 0f and x = –1 both are singular points

Also $(x-0)\ P(x)=\frac{x-1}{x(x+1)}$

$(x-0)^2\ Q(x)=\frac{2}{(x+1)^2}$

⇒ p(x) is not analytic at x = 0

⇒ x = 0 is an irregular singular point

$(x+1)\ P(x)=\frac{(x-1)}{x^2}$

$(x+1)^2\ Q(x)=\frac{2}{x^2}$

⇒ both (x + 1) P(x) and $(x+1)^2$ Q(x) are analytic at x = –1

⇒ x = –1 is regular singular point